Correlation functions and spectra for fields with highly anticorrelated phase jumps

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A non-Markovian jump model of phase-fluctuating electromagnetic fields is considered, which is a generalization of the random phase telegraph model. According to the present model, the phase jumps are highly anticorrelated Markovian jump process, i.e., they have alternating signs and *approximately* equal magnitudes. Closed analytical results for the field correlation function and spectrum have been obtained under rather general assumptions. The parameter γ, which is a measure of correlation of successive jumps, has been expressed in a general form through the parameters of the model. Two regimes have been found, depending on the values of the rms phase jump B and the correlation parameter γ. For highly anticorrelated and not very large jumps ($1+\gamma \ll 1, B^{-2}$) the correlation function (spectrum) consists of two components with significantly different damping rates (widths). In the opposite case each phase jump destroys the coherence of the field and the decay rate of the correlation function is determined by the frequency of jumps. The possibility of extracting parameters of the model from measurements of the laser field correlation function and/or spectrum is discussed.

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I. INTRODUCTION

Statistical properties of laser radiation are of great importance in such fields as nonlinear optics and optical communications. A number of models were introduced to describe a fluctuating laser field and its interaction with matter. In the case of an intensity stabilized, phase-fluctuating field, which is of interest here, the theories suggested included the models of phase diffusion [1–7], Markovian phase jumps [8–10], Gaussian-Markovian frequency fluctuations [11–15], and phase telegraph (dichotomic) noise [16–20]. In Ref. [21] (hereafter referred to as I) the generalized jump model (GJM) of phase-fluctuating laser fields was introduced. According to the GJM, the laser phase jumps randomly at random moments of time, the successive jump values being statistically dependent (correlated). The GJM was shown in paper I to reduce to all the above models in the respective regions of the B,γ space, where B is the root-mean-square jump and γ is the correlation parameter. For the GJM the laser field correlation function and spectrum were obtained analytically in almost all cases of interest in the general form (paper I). In Refs. [22] and [23] the GJM was applied to study the effect of phase fluctuations on the resonance fluorescence spectrum. A special consideration was given in Refs. [21–23] to the generic case of the Keilson-Storer model (KSM) which allowed one to obtain an especially simple formula for all cases of interest. In Ref. [24] an exact expression for the field correlation function in the KSM case was obtained. Monte Carlo simulations were used in Refs. [21–24] to verify the results of the theory for the KSM case.

It is worth noticing the difference between the phase telegraph model (PTM) and other theories of phase fluctuations. In the PTM the phase varies in a limited domain (consisting of two values for the case in question), so the field dephasing is generally not complete and the correlation function tends to a finite value at infinite time. In contrast to this, the other models mentioned above assume that the phase performs a kind of Brownian motion and increases steadily on the average. This yields the complete dephasing of the field on the time scale of a certain correlation time. Correspondingly, these models yield the field spectrum as a bell-like shape with width of the order of the reciprocal correlation time, whereas the PTM field spectrum involves a δ function along with a finite width component. In the framework of the GJM, the PTM is realized when the phase jumps are completely anitcorrelated ($\gamma=-1$). The GJM allows one, in principle, to consider any value of γ from the allowed interval $-1 \leq \gamma \leq 1$, including the case of highly anticorrelated jumps ($\gamma \approx -1$), where the field can be expected to behave very similarly to the PTM case. In paper I the domain of highly correlation jumps was studied only for the special case of the KSM, whereas all the other limits were considered in the general form.

Measurements of the correlation function and the spectrum of the laser field provide important information on the field statistical properties [25]. In the present paper the correlation function and the spectrum are calculated for a phase-fluctuating field described by an anticorrelated random jump process of a general form. Closed analytical results are obtained. In particular, the results obtained allow one to trace the transformation of the spectrum with the change of B and γ, including a continuous transition between a PTM-like spectrum, consisting of two components of different widths, and one-component spectra characteristic of other models of phase fluctuations.

In Sec. II the generalized jump model of phase fluctuations is reviewed. The field correlation function and spectrum are obtained in Secs. III and IV for the cases of fully and highly anticorrelated jumps, respectively. Section V provides concluding remarks. The Appendixes provide

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supplementary material used in the main text. Appendix A discusses the quantitative characterization of non-Lorentzian line shapes and nonexponential decay laws typical of the present model, in Appendix B highly anticorrelated Markovian jump processes are studied, Appendix C provides relevant information on cumulant expansions, and Appendix D contains details of the calculations.

II. THE GENERALIZED JUMP MODEL OF PHASE FLUCTUATIONS

A. The description of the model

Consider a laser field of the form

$$E(t)=E_0 e^{-i[\omega_L t+\phi(t)]}+\text{c.c.}=\tilde{E}(t)e^{-i\omega_L t}+\text{c.c.}\ , \quad (2.1)$$

where ω_L is the nominal frequency, E_0 is the field amplitude which is assumed to be constant, $\phi(t)$ is a fluctuating phase which is described by the generalized jump model, and $\tilde{E}(t)$ is the complex field amplitude. The GJM is defined as follows (see the details in paper I). The laser phase $\phi(t)$ abruptly changes by values β_k at random moments t_k. If, e.g., there are n phase jumps $\beta_1,\ldots,\beta_n$ between 0 and t, then

$$\phi(t)=\phi(0)+\beta_1+\cdots+\beta_n \quad (0<t_1<\cdots t_n<t)\ . \quad (2.2)$$

The phase $\phi(t)$ is a functional of the random function $\beta(t)$, which equals the size of the latest phase change preceding the current moment t. $\beta(t)$ is a Markovian jump process defined by three functions: $\tau_0(\beta)$ is the mean time between successive phase jumps depending generally on the current value of β, $f(\beta)$ is the distribution of the jumps, and $h(\beta',\beta)$ is the conditional probability density for the phase to jump by β if the previous jump was β'. These functions are assumed to obey the following conditions.

(i) The functions are symmetric: $f(\beta)=f(-\beta)$, $\tau_0(\beta)=\tau_0(-\beta)$, and $h(\beta',\beta)=h(-\beta',-\beta)$. (ii) The distribution $f(\beta)$ has a bell-like shape and the ratios of successive even moments of $f(\beta)$ are of the same order, $\langle\beta^{2k}\rangle/\langle\beta^{2k-2}\rangle\sim B^2$ $(k=1,2,\ldots)$, where $B=\langle\beta^2\rangle^{1/2}$. Here and below the average over β is given by

$$\langle\cdots\rangle=\int\cdots f(\beta)d\beta\ . \quad (2.3)$$

(iii) The function $\tau_0(\beta)$ is slowly varying, i.e., its characteristic length of change is of order B or more. (iv) In the case of highly anticorrelated jumps $h(\beta',\beta)$, rewritten as $h(\beta';\alpha)$ with $\alpha=\beta+\beta'$, has a characteristic length of change of order B as a function of β' and a much smaller characteristic length as a function of α. This assumption is satisifed, in particular, by the KSM function (2.5). The above rather general conditions allow one to obtain closed expressions for the field correlation function and spectrum, as shown below.

An alternative, more qualitative, characterization of a jump process is by means of three parameters (paper I). One of them is the characteristic phase jump B. Another parameter is the characteristic time interval between successive jumps τ_{ch} equal to a suitable average of $\tau_0(\beta)$. Actually, any average of $\tau_0(\beta)$ with a positive slowly varying weight can serve as the definition of τ_{ch}, since, in view of the above conditions (ii) and (iii), all such averages are of the same order. Here, as in paper I, τ_{ch} is assumed equal to $\tau_{av}=B^2\langle\beta^2/\tau_0(\beta)\rangle^{-1}$. The third parameter is the correlation parameter γ which characterizes the degree of correlation between successive jumps $(-1\le\gamma\le1)$. To illustrate the physical meaning of γ consider the case of the Keilson-Storer model:

$$f(\beta)=\frac{1}{(2\pi B^2)^{1/2}}\exp\left[-\frac{\beta^2}{2B^2}\right]\ ,\quad \tau_0(\beta)=\tau_0\ , \quad (2.4)$$

and

$$h(\beta',\beta)=h(\beta-\gamma\beta')=\frac{1}{[2\pi(1-\gamma^2)B^2]^{1/2}}\exp\left[-\frac{(\beta-\gamma\beta')^2}{2(1-\gamma^2)B^2}\right]\quad (-1\le\gamma\le1)\ . \quad (2.5)$$

As follows from the KSM, the average value of the phase jump β following the jump β',

$$\bar{\beta}_{\beta'}=\int d\beta\, h(\beta',\beta)\beta\ , \quad (2.6)$$

is given by $\gamma\beta'$, leading to the obvious meaning of positive, negative, and zero correlation. For the KSM γ is a parameter of the theory, but, as shown in paper I, the correlation parameter γ can be defined for the most general case [Eq. (3.4) in paper I]. The price of the generality is that the definition of γ in I, though it leads to physically reasonable results, is not transparent and requires a solution of an integro-differential equation [Eq. (2.5) in paper I]. In the present case of highly anticorrelated jumps, $\gamma\approx-1$, an expression for γ much simpler than that in paper I can be obtained (see Sec. IV). The conditional probability and other properties of the Markovian process $\beta(t)$ in the case $\gamma\approx-1$ are discussed in Appendix B.

B. The field correlation function and spectrum

The correlation function is given by

$$k(t)=\langle\tilde{E}(t)\tilde{E}^*(0)\rangle/E_0^2=\langle e^{-i[\phi(t)-\phi(0)]}\rangle\ . \quad (2.7)$$

A one-sided Fourier transform of $k(t)$ yields the laser intensity spectrum normalized to 1,

$$J(\omega)=\frac{1}{\pi}\text{Re}\int_0^\infty k(t)e^{i\omega t}dt\ , \quad (2.8)$$

where ω is the frequency counted off from ω_L. As shown in paper I, for the present GJM

$$k(t)=\int r(\beta,t)d\beta\ , \quad (2.9)$$

where the partially averaged correlation function $r(\beta,t)$ is the solution of the equation

$$\dot{r}=-\frac{r}{\tau_0(\beta)}+e^{-i\beta}\int r(\beta',t)\frac{h(\beta',\beta)}{\tau_0(\beta')}d\beta'\ , \quad (2.10)$$

with the initial condition $r(\beta,0)=f(\beta)$.

III. FULLY ANTICORRELATED JUMPS (THE GENERALIZED TELEGRAPH MODEL)

According to the PTM, the field phase $\phi(t) = \bar{\phi} + \beta(t)/2$, where $\bar{\phi}$ is a random constant and $\beta(t)$ is a random telegraph process which acquires two values, $\pm\beta_0$, with equal probabilities. A simple extension of the PTM is to consider an ensemble of independent telegraph processes with different values of β_0 characterized by a distribution $g(\beta_0)$ ($\beta_0 \geq 0$). According to this generalized telegraph model (GTM) $\beta(t)$ is a Markovian process with the same conditional probability as the telegraph noise, but with a different distribution of β. For the telegraph noise the distribution of β is

$$f_T(\beta) = [\delta(\beta-\beta_0) + \delta(\beta+\beta_0)]/2 ,$$

whereas for the GTM it is given by

$$f(\beta) = \int_0^\infty f_T(\beta) g(\beta_0) d\beta_0 = \tfrac{1}{2} g(|\beta|) . \qquad (3.1)$$

If $g(|\beta|)$ decreases monotonically with $|\beta|$, its maximum being at $\beta=0$, then $f(\beta)$ is a bell-shaped function. This is precisely the case considered in this paper [cf. the condition (ii) in Sec. II A].

The GTM is obtainable as the limit of fully anticorrelated jumps in the GJM ($\gamma = -1$). In this limit $h(\beta',\beta) = \delta(\beta'+\beta)$ and Eqs. (2.8)–(2.10) yield (paper I)

$$k(t) = \langle e^{-2t/\tau_0(\beta)} \sin^2(\beta/2) \rangle + \langle \cos^2(\beta/2) \rangle \qquad (3.2)$$

and

$$J(\omega) = \frac{1}{2\pi} \left\langle \frac{\sin^2(\beta/2)\tau_0(\beta)}{[\omega\tau_0(\beta)/2]^2 + 1} \right\rangle + \langle \cos^2(\beta/2) \rangle \delta(\omega) \qquad (3.3)$$

[cf. (2.3)]. Note that Eqs. (3.2) and (3.3) are obtainable also from the averages over β of the respective results of the PTM [16]. A remarkable feature of the results (3.2) and (3.3) is a nonzero infinite-time limit of the correlation function and, accordingly, a δ-function component of the spectrum. This is explained by the fact that the first term in the expression for the field amplitude

$$\tilde{E}(t) = e^{i\bar{\phi}} \{ \cos[\beta(t)/2] - i \sin[\beta(t)/2] \}$$

is an even function of the phase jump β and thus not affected by the noise.

The areas under the broad and narrow components in the normalized field spectrum are $\langle \sin^2(\beta/2) \rangle$ and $\langle \cos^2(\beta/2) \rangle$, respectively. In the case of small jumps, $B^2 \ll 4$, the area under the broad component is $B^2/4$, being much smaller than the area under the narrow component. The difference between the two areas decreases with the increase of B, and for $B^2 \gg 4$ the areas are equal. For a constant $\tau_0(\beta) = \tau_0$ the time-dependent component of $k(t)$ is exponential and the broad component of the spectrum is Lorentzian with width $2/\tau_0$. In the general case the time-dependent component of $k(t)$ decays with a decreasing rate. The broad component of the spectrum is a superposition of Lorentzians centered at ω_L with different widths.

The above results, as well as those to be obtained below, involve generally nonexponential decay laws and non-Lorentzian line shapes. To characterize such functions two readily measurable parameters are introduced in Appendix A, the characteristic and effective widths. The two parameters, which generally differ from each other, reduce to the half width at half maximum (HWHM) in the case of a Lorentzian line. In particular, the effective width of the broad component of the spectrum (3.3) is obtained from Eqs. (3.2) and (A1a) to be

$$\nu_1^* = \frac{2\langle \sin^2(\beta/2)/\tau_0(\beta) \rangle}{\langle \sin^2(\beta/2) \rangle} \qquad (3.4a)$$

$$\approx \begin{cases} 2/\tau_{av} , & B^2 \ll 4 \\ \langle 2/\tau_0(\beta) \rangle \equiv 2/\bar{\tau} , & B^2 \gg 4 . \end{cases} \qquad (3.4b)$$

Under the above assumptions (i)–(iii) on $f(\beta)$ and $\tau_0(\beta)$, both limiting values of ν_1^* are of the same order of magnitude. The characteristic linewidth (A1b) can be shown to be of the same order as ν_1^*. Thus the linewidth of the broad component of the spectrum does not change significantly with B, being of the order of $2/\tau_{av}$.

IV. HIGHLY ANTICORRELATED JUMPS

If phase jumps are highly, but not fully, anticorrelated, the field phase $\phi(t)$ fluctuates similarly to the random telegraph process. However, now $|\beta(t)|$ is not constant, but undergoes a slow diffusion motion (see Appendix B). The phase $\phi(t)$ (2.2) now varies without bound. This yields complete field dephasing after a sufficiently long time and accordingly the zero limit at the infinite time for the correlation function. Below in this section the problem is studied in two overlapping regions, $(1+\gamma) \ll 1, B^{-2}$ and $(1+\gamma) \ll 1 \ll B$, which together cover the whole region of highly anticorrelated jumps, $1+\gamma \ll 1$.

For highly anticorrelated and not too large jumps, i.e., for $(1+\gamma) \ll 1, B^{-2}$ (cf. Sec. IV C), the correlation function can deviate significantly from Eq. (3.2) found in the case $\gamma = -1$ only for sufficiently long times, $t \gg \tau_{av}$. The analysis of Eq. (2.10) made previously (paper I, Sec. IV E 2) showed that for $t \lesssim \tau_{av}$ indeed Eq. (3.2) approximates the correlation function, whereas for $t \gg \tau_{av}$ the correlation function is given by

$$k(t) = \int R(\beta,t) e^{-i\beta/2} d\beta \quad (t \gg \tau_{av}) , \qquad (4.1)$$

where $R(\beta,t)$ obeys the equation

$$\dot{R} = [-a_2(\beta)/4 + \tilde{L}_\beta] R . \qquad (4.2)$$

Here

$$\tilde{L}_\beta R = \frac{\partial}{\partial\beta}[a_1(\beta)R] + \frac{\partial^2}{\partial\beta^2}[a_2(\beta)R] , \qquad (4.3)$$

where

$$a_n(\beta) = \delta_n(\beta)/[n!\tau_0(\beta)] , \quad \delta_n(\beta) = \int d\alpha\, \alpha^n h(\beta;\alpha) . \qquad (4.4)$$

Equations (4.3) and (4.4) were derived in paper I with the use of the above conditions (i)–(iv). The condition (i) implies also that

$$a_n(\beta)=(-1)^n a_n(-\beta) . \quad (4.5)$$

The matching condition between the short- and long-time asymptotics yields the following initial condition for Eq. (4.2):

$$R(\beta,0)=f(\beta)\cos(\beta/2) . \quad (4.6)$$

Equations (4.6) and (4.5) imply that $R(\beta,t)=R(-\beta,t)$, which in turn allows one to recast (4.1) as

$$k(t)=\int R(\beta,t)\cos(\beta/2)d\beta \quad (t\gg\tau_{\text{av}}) . \quad (4.7)$$

Equation (4.2) is a stochastic differential equation. The first term in the right-hand side of Eq. (4.2) provides the instantaneous field dephasing rate for the realizations of the random field with a given β at t. The second term takes into account fluctuations of $\beta(t)$, which on a time scale greater than τ_{av} have the form of diffusion in the β space (cf. Appendix B). There is a statistical dependence between the initial condition (4.6) and the dephasing rate $a_2(\beta)$, which is seen from the fact that (4.6) does not mimic $f(\beta)$. This results from the β-dependent field dephasing at the first, fast stage. Equation (4.2) cannot be solved analytically for the general case. In the remainder of this section simple solutions for important special cases will be obtained.

A. Small jumps ($B^2\ll 4$)

We will seek $k(t)$ with accuracy up to terms of order B^2. To this end, we solve Eq. (4.2) with the help of the cumulant expansion technique which allows for a statistical dependence between the stochastic differential equation and the initial condition [27] (for details, see Appendix C). Let us cast $R(\beta,t)$ as

$$R(\beta,t)=\bar{R}(t)f(\beta)+\delta R(\beta,t) , \quad (4.8)$$

where $\bar{R}(t)=\int R(\beta,t)d\beta$. Inserting Eq. (4.8) into Eq. (4.7) yields

$$k(t)\approx\langle\cos(\beta/2)\rangle\bar{R}(t)-\int(\beta^2/8)\delta R(\beta,t)d\beta \approx(1-B^2/8)\bar{R}(t) . \quad (4.9)$$

Here we took into account that $\int\delta R(\beta,t)d\beta=0$, which results from the integration of both sides of Eq. (4.8) over β, and used the relation $\int\beta^2\delta R(\beta,t)d\beta\sim B^4\bar{R}(t)$ (Appendix C). As shown in Appendix C, for $B^2\ll 4$, up to the second order in B $\bar{R}(t)$ obeys the equation

$$\dot{\bar{R}}=-\nu_a\bar{R} \quad [\nu_a=\langle a_2(\beta)\rangle/4] \quad (4.10)$$

with the initial condition

$$\bar{R}(0)=\langle\cos(\beta/2)\rangle\approx 1-B^2/8 . \quad (4.11)$$

Finally, an interpolation formula that correctly describes the short- and long-time behavior of the correlation function of the field in the case of small jumps, $B^2\ll 4$, is [cf. Eqs. (3.2) and (4.9)–(4.11)]

$$k(t)=(1-B^2/4)e^{-\nu_a t}+\tfrac{1}{4}\langle\beta^2 e^{-2t/\tau_0(\beta)}\rangle . \quad (4.12)$$

The second term here is both small and fast-decaying, as compared to the first term. For $t\gg\tau_{\text{av}}$ in the lowest order in B $k(t)\approx e^{-\nu_a t}$. In this approximation the field fluctuations are describable by the PDM (cf. paper I) with the phase diffusion coefficient equal to ν_a. The intensity spectrum of the field follows from Eqs. (2.8) and (4.12) as

$$J(\omega)=\frac{1}{\pi}\left\{\frac{(1-B^2/4)\nu_a}{\omega^2+\nu_a^2}+\left\langle\frac{\beta^2\tau_0(\beta)/8}{[\omega\tau_0(\beta)/2]^2+1}\right\rangle\right\} . \quad (4.13)$$

Equation (4.13) shows that the narrow component of the spectrum is a Lorentzian with the HWHM ν_a. The area under the narrow component is much larger than the area under the broad component.

The expression (4.12) for the correlation function can be compared with the previously known result [Eq. (4.23) in paper I]. For the present case, $\gamma\approx-1$, Eq. (4.23) in I and Eq. (4.12) here have the same validity conditions, $B^2\ll 4$, so the equations can be identified (with accuracy up to B^2). The result in I depends on correlation functions of the random process $\beta(t)$, which generally are not known, whereas Eq. (4.12) expresses $k(t)$ explicitly through the parameters of the stochastic model. Compare separately the fast and the slowly decaying terms in the two expressions. The comparison shows that the fast decaying term $q(t)$ in Eq. (4.23) in I can be approximated for $\gamma\approx-1$ by its value at $\gamma=-1$. Indeed, the calculation of $q(t)$ [Eq. (4.21) in I] with the help of Eq. (B5) in I for the function $\theta_0(t)$ valid for $\gamma=-1$ yields a result identical to the second, fast decaying term in Eq. (4.12). This fact is explained by noting that the function $\theta_0(t)$ is expressed in terms of two-time moments of odd functions of β [see Eqs. (4.17), (4.16), and (3.2) in I], which are shown in Appendix B to be close for $\gamma\approx-1$ to their values at $\gamma=-1$. The slowly decaying terms can be identified if

$$\nu_a=\frac{(1+\gamma)B^2}{4\tau_{\text{av}}} . \quad (4.14)$$

The correlation parameter

Comparing the expressions for ν_a (4.10) and (4.14) and using Eq. (4.4) for $a_2(\beta)$ allows one to relate the correlation parameter to the functions that define the process $\beta(t)$,

$$\gamma=(\tau_{\text{av}}/B^2)\int\int d\beta' d\beta f(\beta')h(\beta',\beta)\beta'\beta/\tau_0(\beta') . \quad (4.15)$$

This formula has a clear physical meaning. For the KSM case, Eqs. (2.4) and (2.5), it becomes an identity. For the general case $\bar{\beta}_{\beta'}$ (2.6) is not proportional to β' and a simple way to get a β-independent correlation parameter is to average $\bar{\beta}_{\beta'}$ over β' (on multiplying it by β' to get an even function of β') with the weight given by the rate $1/\tau_0(\beta')$ of the change from β' to another value of $\beta(t)$. On dividing this average by its value for fully correlated jumps, when $h(\beta',\beta)=\delta(\beta-\beta')$, one gets Eq. (4.15). Equation (4.15) has all the properties of the correlation parameter. Indeed, the expression (4.15) assumes the values 1, -1, and 0 for the cases of fully correlated, fully anticorrelated, and uncorrelated [Eq. (4.5) in I] jumps, respectively. For a general case of partially correlated jumps Eq. (4.15) varies between -1 and 1, being positive (negative) if successive values of $\beta(t)$ tend to be of the same (opposite) sign. Equation (4.15) has a much simpler form than the definition of γ [Eq. (3.4) in paper I] given in terms of the two-time moment of the random process $\beta(t)$. Thus, despite the apparent difference between the definitions of γ in Eq. (4.15) and in paper I, their equivalence was proved here for $\gamma \approx -1$.

B. The effective width of the narrow component of the spectrum

When B is not small the slow component of the correlation function is generally nonexponential and, accordingly, the narrow component of the spectrum is non-Lorentzian. The solution cannot be obtained now for the general case [28]. However, one can find intermediate asymptotics of the correlation function and the effective width of the narrow component of the spectrum, as follows. Inserting $R(\beta,t) \approx R(\beta,0)$ [Eq. (4.6)] into the right-hand side of Eq. (4.2) and substituting the resulting solution of Eq. (4.2) into Eq. (4.7), taking account of Eq. (B12b) and the equality $\langle a_1(\beta)\sin\beta\rangle = \langle a_2(\beta)\cos\beta\rangle$, which follows from the identity (B13), one gets $k(t) = \langle \cos^2(\beta/2)\rangle(1-\nu_2^* t)$ for $\tau_{\rm av}/2 \ll t \ll 1/\nu_a$. Here

$$\nu_2^* = \nu_a / \langle \cos^2(\beta/2)\rangle \approx \begin{cases} \nu_a\ , & B^2 \ll 4 \qquad (4.16a)\\ 2\nu_a\ , & B \gg 1\ . \qquad (4.16b)\end{cases}$$

In particular, for a Gaussian $f(\beta)$ (2.4) Eq. (4.16) yields $\nu_2^* = 2\nu_a/(1+e^{-B^2/2})$. The formula which describes the correlation function for short and intermediate times is [cf. Eq. (3.2)]

$$k(t) = \langle \sin^2(\beta/2) e^{-2t/\tau_0(\beta)}\rangle + \langle \cos^2(\beta/2)\rangle(1-\nu_2^* t) \quad (\nu_a t \ll 1)\ . \tag{4.17}$$

For long times, $\nu_a t \gtrsim 1$, the correlation function generally decays with a decreasing damping rate which is on the average of order ν_a.

The narrow component of the laser spectrum has the integral intensity $\langle \cos^2(\beta/2)\rangle$ and the effective width ν_2^*. The Fourier transform of Eq. (4.17) yields the following formula for the wings of the spectrum:

$$J(\omega) = \frac{\nu_a}{\pi\omega^2} + \frac{1}{2\pi}\left\langle \frac{\sin^2(\beta/2)\tau_0(\beta)}{[\omega\tau_0(\beta)/2]^2+1}\right\rangle \quad (|\omega| \gg \nu_a)\ . \tag{4.18}$$

C. Large jumps ($B \gg 1$)

As shown above, for large jumps the effective widths of the broad and narrow components of the field spectrum are $2/\bar{\tau} \sim 2/\tau_{\rm av}$ and $2\nu_a$, respectively [cf. Eqs. (3.4b) and (4.16b)]. The narrow component width relative to the broad one is of order $(1+\gamma)B^2$. When the jump size is so large that $(1+\gamma)B^2 \gtrsim 1$ the field dephasing process cannot be separated into fast and slow stages and Eqs. (4.1) and (4.2) do not hold. To consider the case of large jumps irrespective of the value of $(1+\gamma)B^2$ we proceed from the exact Eq. (2.10).

Let $Q(\beta_0,\beta,t)$ be a solution of Eq. (2.10) with the initial condition $Q(\beta_0,\beta,0) = \delta(\beta-\beta_0)$. Then Eq. (2.9) for $k(t)$ can be rewritten as

$$k(t) = \int\int d\beta_0 d\beta\, f(\beta_0) Q(\beta_0,\beta,t)\ . \tag{4.19}$$

For $t \ll 1/\nu'_\beta = \tau_{\rm av}/(1+\gamma)$ $Q(\beta_0,\beta,t)$ can be cast as

$$Q(\beta_0,\beta,t) = Q_+(\beta,t) + Q_-(\beta,t)\ , \tag{4.20}$$

where $Q_+(\beta,t)$ [$Q_-(\beta,t)$] as a function of β vanishes outside a small vicinity of β_0 ($-\beta_0$). The two vicinities do not overlap, except for very small $|\beta_0|$ which make an insignificant contribution into the integral (4.19). Inserting (4.20) into Eq. (2.10) and taking account of the conditions (iii) and (iv) in Sec. II A yields a system of 2 equations,

$$\dot{Q}_+(\beta,t) = -[Q_+(\beta,t) - e^{-i\beta}\int h(\beta_0;\beta'-\beta) Q_-(-\beta',t) d\beta']/\tau_0(\beta_0)\ , \tag{4.21a}$$

$$\dot{Q}_-(\beta,t) = -[Q_-(-\beta,t) - e^{i\beta}\int h(\beta_0;\beta'-\beta) Q_+(\beta',t) d\beta']/\tau_0(\beta_0)\ , \tag{4.21b}$$

with the initial conditions $Q_+(\beta,0) = \delta(\beta-\beta_0)$ and $Q_-(\beta,0) = 0$. The solution of Eqs. (4.21) yields (Appendix D)

$$k(t) = \langle e^{-t/\tau_0(\beta)} \cosh[\tilde{H}(\beta) t/\tau_0(\beta)]\rangle\ , \tag{4.22}$$

where

$$\tilde{H}(\beta) = \left[\int h(\beta;\alpha) e^{i\alpha} d\alpha\right]^{1/2}\ . \tag{4.23}$$

Consider two limiting cases of Eq. (4.22).

For moderately large jumps, $1 \ll B^2 \ll (1+\gamma)^{-1}$, one gets

$$\tilde{H}(\beta) \approx [1 + i\delta_1(\beta) - \delta_2(\beta)/2]^{1/2} \approx 1 - \delta_2(\beta)/4\ .$$

Here $\delta_1(\beta)$ was neglected in comparison with $\delta_2(\beta)$, since

$$\delta_1(\beta) \sim (1+\gamma)B \ll \delta_2(\beta) \sim (1+\gamma)B^2\ .$$

Inserting the above expression for $\tilde{H}(\beta)$ into Eq. (4.22) yields

$$k(t)=\tfrac{1}{2}[\langle e^{-a_2(\beta)t/2}\rangle+\langle e^{-2t/\tau_0(\beta)}\rangle] . \quad (4.24)$$

The correlation function decays to half its initial value at a time of order $\bar{\tau}$ and then tends to zero at a much slower rate of order $(1+\gamma)B^2/(2\tau_{av})$. Both stages of the decay are generally nonexponential with monotonically decreasing decay rates. The spectrum,

$$J(\omega)=\frac{1}{2\pi}\left[\left\langle\frac{a_2(\beta)/2}{\omega^2+a_2^2(\beta)/4}\right\rangle+\left\langle\frac{2/\tau_0(\beta)}{\omega^2+4/\tau_0^2(\beta)}\right\rangle\right] , \quad (4.25)$$

consists of two components of the same integral intensities but significantly different widths. The effective widths of the narrow and broad components (the first and second terms, respectively) are correspondingly $2\nu_a$ and $2/\bar{\tau}$, as mentioned above.

In the very large jump limit, $(1+\gamma)B^2\gg1$, the characteristic width of $h(\beta;\alpha)$ as a function of α is much greater than 1:

$$\delta_2^{1/2}(\beta)=[a_2(\beta)\tau_0(\beta)]^{1/2}\sim(1+\gamma)^{1/2}B\equiv\delta\gg1 .$$

Hence $\tilde{H}(\beta)\approx0$ and Eq. (4.22) becomes

$$k(t)=\langle\exp[-t/\tau_0(\beta)]\rangle . \quad (4.26)$$

Equation (4.26) agrees with Eq. (4.33) in I obtained for the case of large jumps. The spectrum is given by a bell-shaped line with the effective width $1/\bar{\tau}$,

$$J(\omega)=\frac{1}{\pi}\left\langle\frac{1/\tau_0(\beta)}{\omega^2+1/\tau_0^2(\beta)}\right\rangle . \quad (4.27)$$

This line shape is geometrically similar to that of the broad component in Eq. (4.25), the width of (4.27) being equal to half the width of the latter.

The boundary between the above two limits is $(1+\gamma)B^2\sim1$, which can be understood in the following way. The spectrum consists of two components of significantly different widths, if many highly anticorrelated phase jumps are needed to randomize the field phase. Taking into account that $\delta=(1+\gamma)^{1/2}B$ characterizes the change of $|\beta|$ in a collision, the region (called the region E in I) where the correlation function (spectrum) is characterized by two significantly different damping rates (widths) is bounded by the conditions $\gamma\approx-1$ and $(1+\gamma)B^2\ll1$. For small and intermediate highly anticorrelated jumps, $B\lesssim1$, the condition $(1+\gamma)B^2\ll1$ always holds, if $\gamma\approx-1$, but for $B\gg1$ it can fail. If the jumps are so large that $\delta=(1+\gamma)^{1/2}B\gg1$, each phase jump destroys the coherence of the field. Therefore the correlation function (4.26) is independent of the correlation parameter γ and decays with a rate of the order of $1/\tau_{av}$ (the region C [21] or quasiuncorrelated limit [24]).

The transition between the regions C and E is given by Eq. (4.22). The spectrum in the case $B^2\sim(1+\gamma)^{-1}$ is a bell-shaped line with the effective width $1/\bar{\tau}$. To get the results in a more explicit form, consider the example of a Gaussian $h(\beta;\alpha)$,

$$h(\beta;\alpha)=\frac{1}{[2\pi\delta_2(\beta)]^{1/2}}\exp\left\{-\frac{[\alpha-\delta_1(\beta)]^2}{2\delta_2(\beta)}\right\} . \quad (4.28)$$

The assumption (4.28) holds, in particular, for the KSM, but generally Eq. (4.28) does not imply that $f(\beta)$ is Gaussian. Inserting Eq. (4.28) into Eq. (4.23) yields $\tilde{H}(\beta)=\exp[i\delta_1(\beta)-\delta_2(\beta)/2]$. In the first approximation one can neglect $\delta_1(\beta)$ here, on account of the remark before Eq. (4.24). Then Eq. (4.22) yields

$$k(t)=\langle e^{-t/\tau_0(\beta)}\cosh[te^{-\delta_2(\beta)/4}/\tau_0(\beta)]\rangle . \quad (4.29)$$

The result (4.29) readily implies the above two limits, Eqs. (4.24) and (4.26). The spectrum now is

$$J(\omega)=\frac{1}{2\pi}\sum_{j=\pm1}\left\langle\frac{[1+je^{-a_2(\beta)/4}]/\tau_0(\beta)}{\omega^2+[1+je^{-a_2(\beta)/4}]^2/\tau_0^2(\beta)}\right\rangle . \quad (4.30)$$

For the KSM case, where $\tau_0(\beta)=\tau_0$ and $\delta_2(\beta)=2(1+\gamma)B^2$, Eqs. (4.29) and (4.30) simplify. In particular [29],

$$k(t)=e^{-t/\tau_0}\cosh[(t/\tau_0)e^{-(1+\gamma)B^2/2}] . \quad (4.31)$$

D. Discussion

The present theory allows one to consider also the inverse problem of determining the parameters of the theory from measurements of the field corrrelation function and/or spectrum, on the assumption that the present model describes the real situation. Generally speaking, different information can be extracted for different regimes, as follows. In the case of small phase jumps, $B^2\ll4$, the slow decay rate of $k(t)$ or the HWHM width of the narrow component of the spectrum provides the phase diffusion constant $\nu_a=(1+\gamma)B^2/\tau_{av}$ [cf. Eqs. (4.12)–(4.14)]. If, moreover, the jumps are large enough so that the small, fast decaying component of $k(t)$ or the broad, weak component of the spectrum [the second terms in Eqs. (4.12) and (4.13), respectively] can be resolved, one can obtain the three parameters of the model: B, γ, and τ_{av}. Indeed, the amplitude and the initial decay rate of the fast component of $k(t)$ (or the area under and the effective width of the broad component of the spectrum) equal, respectively, $B^2/4$ and $2/\tau_{av}$ [cf. Eqs. (3.2) and (3.4a)], which together with ν_a provide the above three parameters. For moderately large jumps, $1\ll B^2\ll(1+\gamma)^2$, the effective widths of the spectral components [or the initial decay rates of the components of the correlation function (4.24)] provide $2\nu_a$ and $2\bar{\tau}$ [Eqs. (3.4b) and (4.16b)]. In the case of $\tau_0(\beta)=\tau_0=$const the fast component of the correlation function (the broad component of the spectrum) is exponential (Lorentzian), $\bar{\tau}=\tau_{av}=\tau_0$, and one obtains $(1+\gamma)B^2$, in addition to τ_0, but not γ and B separately. For a general $\tau_0(\beta)$ the value $(1+\gamma)B^2$ can be found only by order of magnitude. Finally, for very large jumps, $B^2\gg(1+\gamma)^{-1}$, the correlation function and the spectrum provide the parameters $\nu^*=1/\bar{\tau}$ and $\nu_c=\langle\tau_0(\beta)\rangle^{-1}$ which characterize the jump interval function $\tau_0(\beta)$, but yield no direct information on B or γ.

Thus measurements of the field correlation function or spectrum provide important information on the statistical properties of a given noisy source, but this information may be incomplete. At the same time, the nonlinear effects, such as resonance fluorescence [2,9–11,16,17,19,22,23] and multiphoton resonance phenomena [1,3,5,13,28], are known to be very sensitive to the field statistics. The interesting problem of nonlinear effects induced by a field with highly anticorrelated phase jumps is to be considered elsewhere.

V. CONCLUSION

In the case of highly anticorrelated phase jumps ($\gamma \approx -1$) the random process $\beta(t)$, which underlies the statistical properties of the stochastic field in the GJM, has two significantly different memory times, $\tau_{av}/2$ and $\tau_{av}/(1+\gamma)$. In the present paper the field correlation function and the spectrum were obtained in the general form for all allowed values of the parameters of the problem. Two regimes were found. The two-time-scale nature of fluctuations of $\beta(t)$ gives rise to a peculiar two-component structure of the field correlation function and spectrum in the case of not too large jumps, $(1+\gamma)B^2 \ll 1$. The width of the narrow component of the spectrum is of the order of $(1+\gamma)B^2/\tau_{av}$, increasing with the increase of the characteristic phase jump B and the correlation parameter γ. The width of the broad component is of the order of $2/\tau_{av}$ and the relative weight of the component increases with B. For small B the results of this paper for the field correlation function and spectrum agree with those obtained in paper I. However, the advantage of the present results is that they express the quantities of interest directly in terms of the parameters of the theory, rather than in terms of two-time moments of $\beta(t)$, as in paper I. In particular, an explicit expression for γ was obtained. For very large jumps, $(1+\gamma)B^2 \gtrsim 1$, each phase jump destroys the coherence of the field. Therefore the correlation function and the spectrum are given by one-component expressions independent of γ, the linewidth being of the order of $1/\tau_{av}$. The above peculiar features distinguish the present case from other models of laser noise.

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APPENDIX A: CHARACTERISTIC LINEWIDTHS AND DECAY RATES

If $F(t)$ is a positive, decreasing function tending to zero for $t \to \infty$, its decay rate can be characterized by the parameters ν^* and/or ν_c defined by

$$\nu^* = -\dot{F}(0)/F(0) , \tag{A1a}$$

$$\nu_c = F(0) \Big/ \int_0^\infty F(t) dt . \tag{A1b}$$

Here ν^* and ν_c are the initial and average decay rates, respectively. Consider now the Fourier transform of $F(t)$ of the form (2.8),

$$S(\omega) = \frac{1}{\pi} \mathrm{Re} \int_0^\infty F(t) e^{i\omega t} dt . \tag{A2}$$

The parameters ν^* and ν_c define, respectively, the wings and the height of $S(\omega)$,

$$S(\omega) = A\nu^*/(\pi\omega^2) \quad (|\omega| \gg \nu^*) , \quad S(0) = A/(\pi\nu_c) , \tag{A3}$$

where $A = F(0)$ is the area under the line $S(\omega)$. For a Lorentzian line, ν^* and ν_c are equal to the HWHM. For a non-Lorentzian line ν^* and ν_c characterize the spectral width and can be called the effective and characteristic widths, respectively.

If the decay rate $-\dot{F}/F$ of the function $F(t)$ does not increase with time, as is the case, e.g., for the time-dependent term in Eq. (3.2), then the relation $\nu_c \le \nu^*$ is valid, the equality holding only for an exponential decay and accordingly for a Lorentzian line.

APPENDIX B: CONDITIONAL PROBABILITY FOR HIGHLY ANTICORRELATED JUMP PROCESS $\beta(t)$

The Markovian process $\beta(t)$ is fully defined if the conditional probability $f(\beta_0;\beta,t)$ to obtain β at t, subject to $\beta(0)=\beta_0$, is known. The conditional probability obeys the Kolmogorov-Feller equation [21,26]

$$\frac{\partial f}{\partial t} = -\frac{f}{\tau_0(\beta)} + \int f(\beta_0;\beta',t) \frac{h(\beta',\beta)}{\tau_0(\beta')} d\beta' , \tag{B1}$$

with the initial condition $f(\beta_0;\beta,0) = \delta(\beta-\beta_0)$. As shown below, in the case of highly anticorrelated phase jumps the process $\beta(t)$ has two characteristic times, the correlation time $1/\nu_\beta = \tau_{av}/2$ and the chaotization time $1/\nu'_\beta = (1+\gamma)/\tau_{av}$. Consider now $f(\beta_0;\beta,t)$ at two overlapping intervals, $t \ll 1/\nu'_\beta$ and $t \gg \tau_{av}$.

At $t \ll 1/\nu'_\beta$ the conditional probability is assumed to be a sum of two components, f_+ and f_-,

$$f(\beta_0;\beta,t) = f_+(\beta_0;\beta,t) + f_-(\beta_0;\beta,t) \quad (\nu'_\beta t \ll 1) , \tag{B2}$$

which vanish everywhere, except for the near vicinities of β_0 and $-\beta_0$ respectively. The components as functions of β are assumed not to overlap, which requires that $|\beta_0|$ be much greater than the characteristic jump of $|\beta|$, $\delta = (1+\gamma)^{1/2}B$. Inserting (B2) into Eq. (B1) and making use of the conditions (iii) and (iv) in Sec. II A yields the following system of two equations:

$$\frac{\partial f_\pm}{\partial t} = -\frac{1}{\tau_0(\beta_0)} \left[f_\pm - \int d\beta' h(\mp\beta_0;\beta+\beta') \times f_\mp(\beta_0;\beta',t) \right] . \tag{B3}$$

Solving Eqs. (B3) by Fourier transform with the initial conditions $f_+(\beta_0;\beta,0) = \delta(\beta-\beta_0)$ and $f_-(\beta_0;\beta,0) = 0$ yields

$$f(\beta_0;\beta,t)=\frac{1}{2\pi}\int d\chi\, e^{-i\chi\beta}\left\{\mathrm{Re}\exp\left[i\chi\beta_0-\frac{1-H(\beta_0,\chi)}{\tau_0(\beta_0)}\right]+i\,\mathrm{Im}\exp\left[i\chi\beta_0-\frac{1+H(\beta_0,\chi)}{\tau_0(\beta_0)}t\right]\right\}\quad(\nu'_\beta t\ll 1)\,, \tag{B4}$$

where

$$H(\beta_0,\chi)=\int d\alpha\, h(\beta_0;\alpha)e^{-i\chi\alpha}\,. \tag{B5}$$

Equation (B4) can be simplified by setting

$$H(\beta_0,\chi)\approx 1-i\delta_1(\beta_0)\chi-\delta_2(\beta_0)\chi^2/2$$

in the first term and $H(\beta_0,\chi)\approx 1$ in the second term of the integrand in (B4), which yields

$$f(\beta_0;\beta,t)=\frac{1}{4(\pi a_2 t)^{1/2}}\left\{\exp\left[-\frac{(\beta-\beta_0+a_1 t)^2}{4a_2 t}\right]+\exp\left[-\frac{(\beta+\beta_0-a_1 t)^2}{4a_2 t}\right]\right\}$$
$$+\tfrac{1}{2}[\delta(\beta-\beta_0)-\delta(\beta+\beta_0)]e^{-2t/\tau_0(\beta_0)}\quad(\nu'_\beta t\ll 1)\,, \tag{B6}$$

where $a_n=a_n(\beta_0)$ $(n=1,2)$ [cf. Eq. (4.4)]. This approximation may introduce an error of order $(1+\gamma)^{1/2}B$ into the widths of the components of $f(\beta_0;\beta,t)$, which is of order of or greater than the widths themselves for $t\sim\tau_0(\beta_0)$, but is insignificant for $t\gg\tau_0(\beta_0)$. This error can be neglected for all times when the conditional probability (B6) is used to obtain two-time averages of functions of β_0 and β with characteristic lengths of change much greater than $(1+\gamma)^{1/2}B$.

For $t\gg\tau_{\mathrm{av}}$ a method similar to that applied in [26] to derive the Fokker-Planck equation can be used [cf. the derivation of Eqs. (4.55) in paper I] to obtain from (B1) the equation

$$\frac{\partial f}{\partial t}=\tilde{L}_\beta f\,, \tag{B7}$$

with the initial condition

$$f(\beta_0;\beta,0)=[\delta(\beta-\beta_0)+\delta(\beta+\beta_0)]/2\,. \tag{B8}$$

Equations (B7) and (B8) were derived taking account of Eq. (4.5) and the fact that the conditional probability is an even function of β for $t\gg\tau_0(\beta_0)$ [cf. Eq. (B6)]. One can check that the solution of (B7) approximately equals (B6) in the interval $\tau_{\mathrm{av}}\ll t\ll 1/\nu'_\beta$.

An expression for the conditional probability valid for all $t\ge 0$ can be written taking account of Eqs. (B6)–(B8) as

$$f(\beta_0;\beta,t)=\tfrac{1}{2}\{[\delta(\beta-\beta_0)-\delta(\beta+\beta_0)]e^{-2t/\tau_0(\beta_0)}$$
$$+[G(\beta_0;\beta,t)+G(-\beta_0;\beta,t)]\}\,. \tag{B9}$$

Equation (B7) has the form of the Fokker-Planck equation, therefore one can conclude that the Green function of Eq. (B7) $G(\beta_0;\beta,t)$ tends to $f(\beta)$ with a relaxation time of order

$$B^2/\langle a_2(\beta)\rangle=\tau_{\mathrm{av}}/(1+\gamma)\equiv 1/\nu'_\beta\,.$$

Here we used that

$$\langle a_2(\beta)\rangle=(1+\gamma)B^2/\tau_{\mathrm{av}}\,, \tag{B10}$$

as follows from Eqs. (4.10) and (4.14). Using Eq. (4.5), one can show that

$$G(\beta_0;\beta,t)=G(-\beta_0;-\beta,t)\,. \tag{B11}$$

It follows that the first (second) term in the braces in Eq. (B9) is an odd (even) function of β and does not contribute to two- and multitime averages of even (odd) functions of $\beta(t)$. As a consequence, multitime moments and cumulants of odd (even) functions of $\beta(t)$ have characteristic decay rates of order $2/\tau_{\mathrm{av}}$ $[\nu'_\beta=(1+\gamma)/\tau_{\mathrm{av}}]$. Moreover, since the odd component of the conditional probability (B9) is the same as for the GTM case $(\gamma=-1)$, multitime moments of odd functions of $\beta(t)$ approximate those for the GTM. At the same time, due to Eq. (B11), one can replace the conditional probability $f(\beta_0;\beta,t)$ by $G(\beta_0;\beta,t)$ in calculations of multitime moments of even functions of $\beta(t)$.

Consider now some properties of the operator $\tilde{L}_\beta$. Since $\tilde{L}_\beta$ enters the Fokker-Planck equation (4.3), it should possess the properties of a stochastic operator [cf. Eqs. (2.2) in paper I],

$$\int d\beta\tilde{L}_\beta g(\beta)=0\,, \tag{B12a}$$

$$\tilde{L}_\beta f(\beta)=0\,. \tag{B12b}$$

Equations (B12) can be derived using the definition of $\tilde{L}_\beta$ in (4.3) above and Eq. (2.5b) in paper I. A useful identity can be inferred from Eq. (B12b). Multiplying both sides of Eq. (B12b) by a function of β and integrating over β by parts yields the identity

$$\langle a_1(\beta)g(\beta)\rangle=\langle a_2(\beta)g'(\beta)\rangle\,. \tag{B13}$$

Here $g(\beta)$ is an arbitrary function, such that $|g(\beta)|$ increases not too fast for $|\beta|\to\infty$. Equation (B13), like Eq. (B12b), holds approximately for $1+\gamma\ll 1$. Note, however, that the equality

$$\langle a_1(\beta)\beta\rangle=\langle a_2(\beta)\rangle\,, \tag{B14}$$

which follows from Eq. (B13) for $g(\beta)=\beta$, is exact. This can be shown by inserting the definitions of $a_1(\beta)$ and $a_2(\beta)$ (4.4) into Eq. (B14).

APPENDIX C: THE GENERALIZED CUMULANT EXPANSION FOR ARBITRARY INITIAL CONDITIONS

In paper I (Appendix A) a projection operator technique for solution of stochastic and operator differential equations was reviewed for the case when the initial condition commutes with the projection operator. Here we review briefly the extension of this technique to the case of an arbitrary initial condition [27] and then apply the method to obtain $k(t)$. Consider the equation

$$\dot{x}(V,t)=(A+B_V+L_V)x(V,t)\ , \tag{C1}$$

where A and B_V are matrices in the vector space of which x is an element and B_V and L_V are operators acting on functions of V. L_V is assumed to obey the conditions (i) $\int L_V g(V)dV=0$ for an arbitrary $g(V)$, and (ii) there is a function $f(V)$ for which $L_V f(V)=0$. The initial condition to Eq. (C1) is $x(V,0)=x_0(V)$.

Introducing the projection operator P by $P\cdots=f(V)\int dV\cdots$, one can show that $PL_V=L_VP=0$. The projection operator technique yields for $\bar{x}(t)=\int x(V,t)dV$ the following equation:

$$\dot{\bar{x}}=(A+\bar{B})\bar{x}+\int_0^t K(t-t')\bar{x}(t')dt'+q(t)\ . \tag{C2}$$

Here $\bar{B}=\int B_V f(V)dV$, $K(t)$ is defined in paper I (Appendix A, Sec. 3), and

$$q(t)=\mu_2(t)+\sum_{n=3}^{\infty}\int_0^t dt_{n-2}\cdots\int_0^{t_2}dt_1\mu_n(t,t_{n-2},\ldots,t_1)\ , \tag{C3}$$

where $\mu_n(t_{n-1},\ldots,t_1)$ is obtained from the totally ordered cumulant $\theta_n(t_{n-1},\ldots,t_1,0)$ defined in I (Appendix A, Sec. 3), by the substitution of the rightmost B_V by $x_0(V)$. In particular,

$$\mu_2(t)=\int\int dV\,dV' B_V e^{At}f(V';V,t)x_0(V')-\bar{B}e^{At}\bar{x}(0)\ , \tag{C4}$$

where $f(V';V,t)$ is the solution of the equation $\dot{f}=L_V f$ with the initial condition $f(V';V,0)=\delta(V-V')$ and $\bar{x}(0)$ is the initial condition for Eq. (C2),

$$\bar{x}(0)=\int x_0(V)dV\ . \tag{C5}$$

An expression for $\delta x(V,t)=x(V,t)-\bar{x}(t)f(V)$ may be also of relevance, as in the present paper. The projection operator technique yields in the lowest order

$$\delta x(V,t)=\int_0^t dt' e^{A(t-t')}\times\int dV'[f(V';V,t)B_{V'}f(V')-\bar{B}f(V)]\bar{x}(t')+e^{At}\int dV' f(V';V,t)\delta x(V',0)\ . \tag{C6}$$

Consider now the application of the above technique to obtain the field correlation function $k(t)$ (4.7). Equation (4.2) is identical to Eq. (C1) if $x=R$, $V=\beta$, $A=0$, $B_V=a_2(\beta)$, and $L_V=L_+$. Hence in the lowest order Eqs. (C2) and (C5) yield the equation

$$\dot{\bar{R}}=-\nu_a\bar{R}+\mu_2(t) \tag{C7}$$

with the initial condition (4.11). The validity condition of Eq. (C7) is $b\tau_c \ll 1$ [Eq. (A16) in paper I], where b is the characteristic perturbation amplitude and τ_c is the cumulant decay time. One gets $b=\langle a_2(\beta)\rangle$, due to the condition (ii) in Sec. II A, whereas $\tau_c=(1+\gamma)/\tau_{\rm av}$, since $a_2(\beta)$ is an even function of β [see Eq. (4.5) and remark after Eq. (B11)]. Using Eq. (B10), the validity condition of Eq. (C7) finally becomes $B^2 \ll 4$.

Show now that $\mu_2(t)$ can be omitted in Eq. (C7). In the lowest order in B Eq. (C4) yields

$$\mu_2(t)=\frac{1}{32}\left[B^2\langle a_2(\beta)\rangle-\int\int d\beta_0 d\beta f(\beta)G(\beta_0,\beta,t)a_2(\beta)\beta_0^2\right]\ . \tag{C8}$$

The effect of this term in Eq. (C7) can be shown to reduce approximately to the multiplication of the solution of Eq. (4.10) by $1+\int_0^t\mu_2(t')dt'/\bar{R}(0)$. The latter term is less than or of order

$$\int_0^\infty \mu_2(t')dt' \sim B^2\langle a_2(\beta)\rangle/(32\nu'_\beta)\sim B^4/32\ . \tag{C9}$$

This exceeds the accuracy of the present approximation for $k(t)$. Hence $\mu_2(t)$ can be omitted in Eq. (C7), yielding Eq. (4.10).

Consider now $\delta R(\beta,t)$. Equation (C6) yields

$$\delta R(\beta,t)=\frac{1}{4}\int_0^\infty\left[\int d\beta' G(\beta',\beta,t-t')a_2(\beta')f(\beta')-\langle a_2(\beta)\rangle f(\beta)\right]\bar{R}(t')dt'+\frac{1}{8}\int f(\beta_0)(\beta_0^2-B^2)G(\beta_0,\beta,t)d\beta_0\ . \tag{C10}$$

Hence

$$\int\beta^2\delta R(\beta,t)d\beta=\frac{1}{4}\int_0^t dt'\{\langle\beta^2(t-t')a_2[\beta(t)]\rangle-B^2\langle a_2(\beta)\rangle\}\bar{R}(t')+\frac{1}{8}[\langle\beta^2(t)\beta^2(0)\rangle-B^4]\ . \tag{C11}$$

Here we took into account that in two-time averages of even functions of β the functions $f(\beta_0;\beta,t)$ and $G(\beta_0,\beta,t)$ are interchangeable (see Appendix B). Noticing that the expressions in the brackets in Eq. (C11) vanish for $\nu'_\beta t \ll 1$ and using Eq. (B10), one can estimate that $\int\beta^2\delta R(\beta,t)d\beta\sim B^4\bar{R}(t)$.

APPENDIX D: DERIVATION OF EQ. (4.22) FOR $k(t)$

Multiplying both sides of Eq. (4.21b) by $e^{-i\beta}$ and Fourier transforming Eqs. (4.21) yields the following equations:

$$\dot{U}_+(\chi,t) = -[U_+(\chi,t) - H(\beta_0,\chi-1)U_-(1-\chi,t)]/\tau_0(\beta_0) ,$$
$$\dot{U}_-(1-\chi,t) = -[U_-(1-\chi,t) - H(\beta_0,\chi)U_+(\chi,t)]/\tau_0(\beta_0) , \quad \text{(D1)}$$

for the components $U_\pm(\chi,t)$ of the Fourier transform of $Q(\beta_0,\beta,t)$ (4.20),

$$U(\beta_0,\chi,t) = \int Q(\beta_0,\beta,t)e^{i\chi\beta}d\beta = U_+(\chi,t)+U_-(\chi,t) . \quad \text{(D2)}$$

In Eqs. (D1) $H(\beta_0,\chi)$ is given by Eq. (B5). The solution of Eqs. (D1) with the initial conditions $U_+(\chi,0) = \exp(i\chi\beta_0)$ and $U_-(\chi,0)=0$ yields,

$$U(\beta_0,\chi,t) = e^{-t/\tau_0(\beta_0)}\left[e^{i\chi\beta_0}\cosh\frac{(H_0H_1)^{1/2}}{\tau_0(\beta_0)}t + (H_1^*/H_0^*)^{1/2}e^{i(1-\chi)\beta_0} \times \sinh\frac{(H_0^*H_1^*)^{1/2}}{\tau_0(\beta_0)}t \right] . \quad \text{(D3)}$$

Here $H_n = H(\beta_0,\chi-n)$ $(n=0,1)$ and the relation $H^*(\beta_0,\chi) = H(\beta_0,-\chi)$ was taken into account. Inserting Eq. (D3) into the expression $k(t) = \langle U(\beta,0,t)\rangle$, which follows from Eqs. (4.19) and (D2), yields finally Eq. (4.22) for the correlation function of the field. To derive Eq. (4.22) we neglected the second term in the square brackets in Eq. (D3) which fast oscillates with β_0 and therefore yields a negligible contribution to $k(t)$.

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